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ORIGINAL PAPER

# An optimistic search equilibrium

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**Abstract** We study a market search equilibrium with aggregate uncertainty, private information and heterogeneous beliefs that are initially optimistic. Despite these biased beliefs, it is shown that all *optimistic equilibria* converge to perfect competition in the limit as the time between matches tends to 0.

**Keywords** Markets with search frictions · Heterogeneous beliefs · Optimism · Bargaining · Aggregate uncertainty

**JEL Classification** C73 · C78 · D83

## 1 Introduction

In this paper, we introduce and study an optimistic search equilibrium. We propose to examine price formation in decentralized markets with *aggregate uncertainty*, an important feature of many markets.[1] Our paper continues the line of research initiated

[1] For example, see Rogerson et al. (2005) for a discussion of its relevance in labor markets.

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in a seminal paper by Yildiz (2003) and followed by Yildiz (2004) and Thanassoulis (2010), in which heterogeneous optimistic beliefs are introduced in a bargaining framework without private information.

The model is based on Mortensen and Wright (2002), but with private information, as in Satterthwaite and Shneyerov (2007, 2008), Shneyerov and Wong (2010a, b, 2011). We consider the simplest form of aggregate uncertainty: even though traders know their own willingness to pay for the good (have *private values*), initially they don't know the aggregate market demand and supply. As a consequence, the traders need to learn at what prices to trade through their market experiences.

We assume that the aggregate uncertainty of the market is represented by a state $\mu \in \{L, H\}$, where $0 < L < H$. This corresponds to the popular press notions of a buyer or seller market. The $H$ and $L$ values of $\mu$ reflect a high or low value of the Walrasian price $p_W(\mu)$, with $p_W(L) < p_W(H)$.

In our model, traders, even initially, have *heterogeneous beliefs*. Regardless of the true state $\mu$, entering buyers believe that they are in the buyer market, $\mu = L$, while entering sellers believe they are in the seller market, $\mu = H$. In other words, traders on both sides of the market start out with *optimistic* initial beliefs. Similar to Yildiz (2003, 2004), and Thanassoulis (2010), we also assume that traders are fully aware of the diverging initial beliefs, so they agree to disagree about the state of the market. As they continue in the market, they may conclude that their original beliefs are wrong and switch to *pessimistic* beliefs: the buyers switch to believing that $\mu = H$, while the sellers, to believing $\mu = L$.

How do the traders discover the true state? The crucial element of our learning process is a simple matching technology whereby in each period the shorter side of the market is fully matched, while the longer side is matched with probability less than 1. Our belief updating mechanism then implies that there is excessive entry by the traders who believe in the wrong state. This in turn will lead to an unbalanced market, with wrong believers on the long side. Once not matched, the wrong believers will update their optimistic beliefs, become pessimistic and trade. Thus, in addition to the *negative* same-side search externality, in our model there is also a *positive* same-side externality because, say, additional buyers speed up the learning process of all buyers.

Figure 1 graphically describes this basic story when the period length $\tau$ is very small. Suppose the true state is $\mu = H$, which corresponds to the outward demand curve. Therefore, it is the optimistic buyers who hold the wrong beliefs that the equilibrium price is $p_W(L)$, and the buyers with his private value above $p_W(L)$ will choose to enter the market. Since the true state is in fact $H$, this will result in an excess entry of the optimistic buyers and make the buyers the longer side of the market.

Our analysis focuses on what we call *optimistic equilibria*. In these equilibria, only traders who share the same (true) belief about the state $\mu$, can trade. When the beliefs diverge, the traders are unable to reach mutually acceptable price and are unable to trade. Consequently, optimistic buyers can only trade with pessimistic sellers, and vice versa. When $\tau$ is small, there are incentives in place to support such an equilibrium. For example, buyers who are optimistic will prefer to wait for a pessimistic seller rather than trade with an optimistic seller at a higher price. This is because the cost of waiting will be small relative to the benefit of a lower price. Similar logic applies to

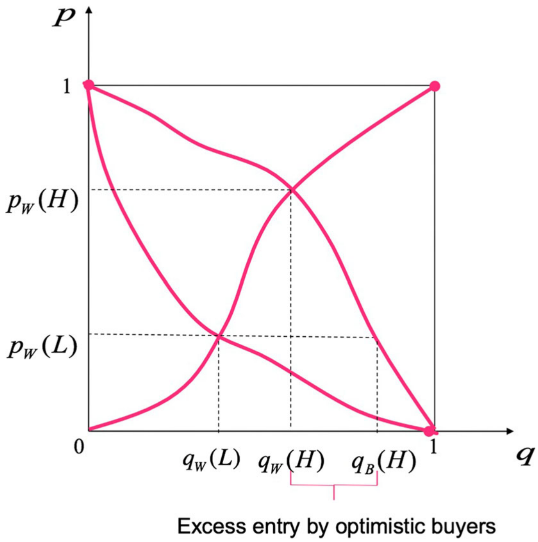



**Fig. 1** Fundamental market imbalance caused by excess entry of optimistic buyers when $\mu = H$

the optimistic sellers. On the other hand, the pessimistic traders will have known the state and will not wait for a deal that they are sure does not exist in the market.

In order to focus on the information transmission and optimism, we restrict our attention to optimistic equilibria with a *full trade* property: every meeting between the traders who share the same belief about the state results in trade. *We show that when both the discount rate and $\tau$ are sufficiently small, there exists a unique optimistic equilibrium.*

Despite the fact that optimistic biases may cause delays and lead to market inefficiency, we show that, in fact, the true state is quickly discovered; even stronger, *the optimistic equilibrium outcomes converge to the efficient ones as frictions disappear, i.e.* $\tau \to 0$.

With small frictions, we show that traders who have a correct belief $\mu$ will propose or accept a price close to $p_W(\mu)$. This means two things. First, the former optimists on the long side with valuations far below (or costs far above) $p_W(\mu)$ will exit. Second, the pessimistic traders on the long side now share the same beliefs with the optimistic traders on the short side and will trade with them. Provided that the state discovery by optimists is quick, their stock in the market is small, and price discovery happens quickly followed by trade (almost) at the right price $p_W(\mu)$. Thus we prove that, as $\tau \to 0$, all optimistic steady-state equilibria converge to the Walrasian outcome in state $\mu$. The traders' utilities converge to their Walrasian counterparts, as if they knew the true state from the beginning. In the limit, there is both full information revelation and efficiency.

This paper is, in spirit, most relevant to the recent bargaining literature with optimistic bargainers initiated by Yildiz (2003). In a follow-up paper Yildiz (2004), he allows players to learn and update their beliefs without restriction as they bargain over time and shows that optimism can cause delays when it is combined with learning. Thanassoulis (2010) also studies a Rubinstein-Stahl bargaining model, in which the players possess heterogeneous optimistic beliefs concerning the probability of arrival of a second buyer. He shows that stalling can occur in equilibrium even if the friction of bargaining tends to zero.

The main difference between our paper and this literature is that we consider a market environment in which buyers and sellers have private information concerning their value or cost and make decisions to enter/stay or exit the market. Traders' decisions to stay or exit endogenously determine the stocks of buyers and sellers, so it is crucial for the optimistic traders to discover the true state when being unmatched. In addition, since in our model matching occurs in each period, learning speeds up when the friction of bargaining gets smaller, so we reach an efficient outcome when $\tau \rightarrow 0$. In Thanassoulis (2010), on the other hand, parties' beliefs are updated continuously with time $t$ through Bayesian updating but do not depend on the period length $\tau$, so stalling still occurs when $\tau \rightarrow 0$

As far as convergence to perfect competition, we are unaware of any published papers that have obtained convergence results under aggregate uncertainty with private-value bilateral bargaining as here.[2] Recently, Lauermann et al. (2012) have also shown convergence to perfect competition in a model with aggregate uncertainty. As in Satterthwaite and Shneyerov (2008), in their model the sellers conduct auctions among the buyers they are matched with. In addition to a different sale mechanism (auctions instead of bilateral bargaining), their model is different in that (i) buyers (sellers) are assumed to be homogeneous in their valuations (costs), and (ii) sellers are assumed to be non-strategic. In their model, the buyers do not know the state of the market, and then learn through unsuccessful bids. Over time, the buyers become more pessimistic and bid more aggressively. It is shown that the equilibrium prices converge to the competitive ones as frictions vanish.

The structure of the paper is as follows. Section 2 introduces the model. Section 3 presents the existence, uniqueness and convergence results. The Appendix contains proofs of some of the results not given in text.

## 2 Model

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period

[2] Several authors have considered steady-state models with *common value* uncertainty, and double auction bargaining with a grid restricted set of price offers. In such a model, Wolinsky (1988) assumes two-sided incomplete information and obtains a negative convergence result, while Serrano and Yosha (1993) assume one-sided incomplete information and show existence of a convergent equilibrium. In addition, Blouin and Serrano (2001) consider a market with one-time entry of agents and obtain strong negative results concerning convergence. However, recently, Gottardi and Serrano (2005) revisit the issue and obtain some positive results in a somewhat different model. In addition, the seminal contributions of Reny and Perry (2006) and Pesendorfer and Swinkels (1997, 2000) provide foundations for a rational expectations equilibria in *static* models of *centralized* double-auction trade with interdependent values. Recently, there has been a renewed interest in common-value information aggregation in financial markets, see e.g. Duffie et al. (2009), Golosov et al. (2009) and Ostrovsky (2009). Most other papers have adopted a private values paradigm with no aggregate uncertainty; a non-exhaustive list includes Butters (1979), Gale (1986, 1987, 2000), Rubinstein and Wolinsky (1985), Wolinsky (1988, 1990), Rubinstein and Wolinsky (1990), McLennan and Sonnenschein (1991), Dagan et al. (1998, 2000), De Fraja and Sakovics (2001), Moreno and Wooders (2002), Serrano (2002), Mortensen and Wright (2002), Satterthwaite and Shneyerov (2007, 2008), Atakan (2009), Lauermann (2009), and Shneyerov and Wong (2010a, b).

$(t = \ldots, -1, 0, 1, \ldots)$ and trade an indivisible, homogeneous good. The length of each period is $\tau$. At the beginning of each period measure $\tau$ of sellers and buyers is born and the newborn traders contemplate entering the market.

The agents in our model are potential buyers and sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller has unit supply. All traders are risk neutral. Potential buyers are heterogeneous in their valuations (or types) $v$ of the good. Potential sellers are also heterogeneous in their costs (or types) $c$ of providing the good. For simplicity, we assume $v, c \in [0, 1]$.

We now introduce the main element of our model, the state of the market $\mu$. The state $\mu$ can take two values, high $(H)$ and low $(L)$ and is drawn by nature once and for all. The $H$ and $L$ values of $\mu$ reflect a high or low value of the Walrasian price $p_W(\mu)$, $p_W(L) < p_W(H)$. Formally, the distributions of valuations $v$ and costs $c$, $G_B(\cdot|\mu)$ and $G_S(\cdot|\mu)$, depend on $\mu$ and the Walrasian price $p_W(\mu)$ is determined through the intersection of the corresponding demand and supply functions:

$$G_S(p_W|\mu) = 1 - G_B(p_W|\mu).$$

Each trader *privately* knows his valuation $v$ if he is a buyer or cost $c$ if he is a seller. We assume that the distributions $G_B(\cdot|\mu)$ and $G_S(\cdot|\mu)$ have densities $g_B(\cdot|\mu)$ and $g_S(\cdot|\mu)$ that are supported on [0, 1] and uniformly bounded from below there,

$$\inf_{v\in[0,1]} g_B(v|\mu) \equiv \underline{g}_B > 0, \quad \inf_{c\in[0,1]} g_S(c|\mu) \equiv \underline{g}_S > 0.$$

We also impose the following standard assumption on the distributions $G_B$ and $G_S$.

**Assumption 1** The Myerson virtual type functions

$$J_B(v|\mu) \equiv v - \frac{1 - G_B(v|\mu)}{g_B(v|\mu)}, \quad J_S(c|\mu) \equiv c + \frac{G_S(c|\mu)}{g_S(c|\mu)}$$

are non-decreasing.

However, traders do not observe $\mu$. The initial beliefs are defined as follows. The buyers' belief is optimistic in that it puts all the weight on $L$, while the sellers' belief is likewise optimistic, in that it puts all the weight on $H$. The prior distribution of $\mu$ is therefore different for buyers and sellers, and there is no common prior over the state.

The instantaneous discount rate is $r \geq 0$, and the corresponding discount factor is $R_\tau = e^{-r\tau}$. Each period consists of the following stages.

1. The mass $\tau$ of potential buyers and sellers are born. Conditional on the true state of the market $\mu \in \{H, L\}$, the new-born buyers draw their valuations $v$ i.i.d. from $G_B(\cdot|\mu)$ and the newborn sellers draw their costs $c$ i.i.d. from $G_S(\cdot|\mu)$.
2. *Participation* (or being active) The new-born potential buyers and sellers decide whether to continue in the market or exit. The exit is permanent. Those who stay, together with the current pools of traders in the market compose the set of active traders. The active buyers and sellers incur participation costs $\tau\kappa$.

3. *Matching* The active buyers and sellers are randomly matched in pairs. The shorter side of the market is matched completely, while the longer side is appropriately rationed. The mass of the matches is given by $\min\{B(\mu), S(\mu)\}$, where $B(\mu)$ and $S(\mu)$ are the steady-state masses of active buyers and active sellers currently in the market. The probability that a buyer is matched is

$$\rho_B(\mu) = \min\left\{1, \frac{S(\mu)}{B(\mu)}\right\},$$

and he is equally likely to meet any active seller. Symmetrically, the seller's matching probability is

$$\rho_S(\mu) = \min\left\{1, \frac{B(\mu)}{S(\mu)}\right\},$$

and she is equally likely to meet any active buyer. The matching is anonymous.
4. *Bargaining* Once a buyer-seller pair is matched, they bargain without observing the type of their partner. The bargaining protocol is *random-proposal take-it-or-leave-it offer*: with probability $1/2$, the seller makes a take-it-or-leave-it offer to the buyer, then the buyer chooses either to accept or reject. And with probability $1/2$, the buyer proposes and the seller responds.[3] We also assume the market is anonymous, so that the bargainers do not know their partners' market history, e.g. how long they have been in the market, what they proposed previously, and what offers they rejected previously.
5. If a type $v$ buyer and a type $c$ seller trade at a price $p$, then they leave the market with payoff $v - p$, and $p - c$ respectively. If bargaining between the matched pair breaks down, both traders will decide if they want to stay or exit the market at the beginning of the next period.

In our market equilibrium construction, there are two logically separate elements, the search equilibrium and the equilibrium of the bargaining game played in a meeting by a buyer and a seller.[4] We describe the search equilibrium first.

## 2.1 Search equilibrium

The market is assumed to be in a steady state. *Heterogeneous beliefs* play a key role in our model. Let $\mu_i$ be the belief of trader $i \in \{B, S\}$ about the true state. In the equilibrium we are describing, $\mu_i$ will only take two values $L$ or $H$. We shall call these *point beliefs*.

The *market continuation values* $W_B(v|\mu_B)$ and $W_S(c|\mu_S)$ are determined from the following recursive Bellman equations,

[3] This is the random-proposer protocol of Rubinstein and Wolinsky (1985). Several papers in the literature have considered other bargaining protocols, notably the k-double auction (k-DA) with grid-restricted price offers. Some references are given below. However, for a k-DA with unrestricted price offers, Shneyerov and Wong (2010a) show existence of non-convergent equilibria even without aggregate uncertainty as here.

[4] This construction parallels that in Satterthwaite and Shneyerov (2007) and Shneyerov and Wong (2010a).

$$W_B(v|\mu_B) = \rho(\mu_B)U_B(v|\mu_B) + R_\tau W_B(v|\mu_B) - \tau\kappa, \tag{1}$$

$$W_S(c|\mu_S) = \rho(\mu_S)U_S(c|\mu_S) + R_\tau W_S(c|\mu_S) - \tau\kappa. \tag{2}$$

where $U_i(\cdot|\mu_i)$ are the expected equilibrium payoffs of the traders in the stage game, which are defined as the payoffs over and above the market continuation values, and $\rho_i(\mu_i)$ is the probability of meeting a trading partner in state $\mu_i$.

The only strategic decision in the search equilibrium is whether or not to engage in search. At the beginning of each period, a trader with belief $\mu_i$ will decide whether to search or exit the market. We denote his *exit* strategy as $\chi_i(\cdot|\mu_i) \in \{0, 1\}$, with 0 indicating exit. A trader will participate in search if and only if his market continuation value at the beginning of the period is nonnegative,

$$\chi_B(v|\mu_B) = \begin{cases} 1, & W_B(v|\mu_B) \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

$$\chi_S(c|\mu_S) = \begin{cases} 1, & W_S(c|\mu_S) \geq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

The exit strategies define the sets of active buyer and seller types in the market, given their beliefs $\mu_i = H, L$:

$$A_B(\mu_B) = \{v : \chi_B(v|\mu_B) = 1\}, \qquad A_S(\mu_S) = \{c : \chi_S(c|\mu_S) = 1\}$$

**Belief updating** Denote the steady-state stock of active buyers who hold a belief $\mu_B$ when the true state is $\mu$ as $B(\mu_B|\mu)$. Likewise, the stock of sellers with a belief $\mu_S$ when the true state is $\mu$ is denoted as $S(\mu_S|\mu)$. Crucial to our equilibrium and results is the *belief updating mechanism*. The newborn traders are assumed to start out *optimistically*, with beliefs $\mu = L$ for the buyers and $\mu = H$ for the sellers. At the beginning of each period, the traders will only update their optimistic beliefs to the pessimistic ones if they did not succeed in meeting a partner in the previous period. Moreover, traders will only update incorrect beliefs. This will happen because the traders with wrong beliefs will be on the long side of the market. This belief updating is formally consistent with the Bayes rule since the beliefs are point beliefs.

We thus have the following condition, which is a part of our equilibrium definition.

**Condition 1 (Fundamental Imbalance Condition)** *The optimists with wrong beliefs about the state are on the long side of the market:*

$$B(H) > S(H), \quad S(L) > B(L).$$

We now turn to the determination of trader stocks $B(\mu_B|\mu)$, $S(\mu_S|\mu)$ and the distributions of active trader types who in state $\mu$ hold the belief $\mu_i$. These distributions are denoted as

$$\Phi_i(\cdot|\mu_i, \mu), \quad i \in \{B, S\}.$$

Further denote as $\hat{q}_B(v|\mu_B, \mu)$ the probability of trading for a buyer when her valuation is $v$, her belief is $\mu_B$ and the true state is $\mu$. The probability $\hat{q}_S(c|\mu_S, \mu)$ is defined similarly. These probabilities are conditional on being matched and are derived from the trading probabilities

$$q_B(v|\mu_S, \mu_B), \quad q_S(c|\mu_B, \mu_S)$$

in the bargaining game. The probability $q_B(v|\mu_S, \mu_B)$ is the subjective probability that a type $v$ buyer who believes the true state is $\mu_B$ will trade conditionally on meeting a seller with a belief $\mu_S$. The probability $q_S(c|\mu_B, \mu_S)$ is defined in parallel. Given the true state $\mu$, the objective probability that a type-$v$ buyer with belief $\mu_B$ will trade, conditionally on meeting a seller, is given by the following expression.

$$\hat{q}_B(v|\mu_B, \mu) = \sum_{\mu_S=H,L} \frac{S(\mu_S|\mu)}{S(\mu)} q_B(v|\mu_S, \mu_B)$$

This expression takes into account the objective probabilities $\frac{S(\mu_S|\mu)}{S(\mu)}$ that, conditionally on being matched, the buyer will meet a seller with belief $\mu_S$ in state $\mu$. In parallel, we have for the sellers

$$\hat{q}_S(c|\mu_S, \mu) = \sum_{\mu_B=H,L} \frac{B(\mu_B|\mu)}{B(\mu)} q_S(c|\mu_B, \mu_S)$$

The trading probabilities $q_B(v|\mu_S, \mu_B)$, and $q_S(c|\mu_B, \mu_S)$ will be defined later, after introducing the bargaining game.

The steady state equations then take the following form. Consider the buyers first. For the optimistic buyers ($\hat{\mu} = L$) who are active in the market, $\chi_B(v|L) = 1$,

$$\tau \cdot dG_B(v|\mu) = \Big(\rho_B(\mu)\hat{q}_B(v|L, \mu) + 1 - \rho_B(\mu)\Big) B(L|\mu) d\Phi_B(v|L, \mu), \tag{5}$$

while for the pessimistic buyers ($\hat{\mu} = H$),

$$\begin{aligned}(1 - \rho_B(\mu))B(L|\mu)d\Phi_B(v|L, \mu) \\ = \Big(\chi_B(v|H)\rho_B(\mu)\hat{q}_B(v|H, \mu) + 1 - \chi_B(v|H)\Big) B(H|\mu)d\Phi_B(v|H, \mu)\end{aligned} \tag{6}$$

Parallel equations apply sellers; for the optimistic ones ($\hat{\mu} = H$) who are active ($\chi_S(c|H) = 1$), we have

$$\tau \cdot dG_S(c|\mu) = \Big(\rho_S(\mu)\hat{q}_S(v|H, \mu) + 1 - \rho_S(\mu)\Big) S(H|\mu) d\Phi_S(c|H, \mu), \tag{7}$$

while for the pessimistic ones, we have

$$\begin{aligned}(1 - \rho_S(\mu))S(H|\mu)d\Phi_S(c|H, \mu) \\ = \Big(\chi_S(c|L)\rho_S(\mu)\hat{q}_S(c|L, \mu) + 1 - \chi_S(c|L)\Big) S(L|\mu)d\Phi_S(c|L, \mu)\end{aligned} \tag{8}$$

Let us now explain the above two equations in detail. For the first Eq. (5) the left-hand side is the per-period mass of buyer types $v$ who enter the market when the true state is $\mu$. The term $B(L|\mu)$ is the stock of optimistic buyers who are in the market in state $\mu$. These are the buyers who initially hold the optimistic belief. The r.h.s. consists of two parts: the mass of optimistic buyer types $v$ who are matched and trade successfully this period and the mass of optimistic buyer types $v$ who are not matched and become pessimistic. The r.h.s. of Eq. (5) therefore reflects the outflow of buyers from the mass of optimistic buyers in each period.

For the second Eq. (6) the l.h.s. is the mass of buyers per-period who have turned pessimistic. Some of these buyers will exit the market as prescribed by the exit strategy $\chi_B(v|H)$. Those who stay, $\chi_B(v|H) = 1$, will only exit through trade, which will happen if, first, they will meet a partner (with probability $\rho_B(H)$), and, second, trading is successful in the bargaining game, which is expected to happen with probability $\hat{q}_B(v|H, \mu)$.

We now formally define the search equilibrium. This definition takes the bargaining equilibrium as *exogenously given*, with the induced expected payoffs $U_i(\cdot|\mu_i)$ and the trading probabilities $q_i(\cdot|\mu_i)$.

**Definition 1** (*Search equilibrium*) The Perfect Bayesian search equilibrium is defined as a tuple

$$\mathcal{S} = \Big(\chi_i(\cdot|\mu_i),\, W_i(\cdot|\mu_i),\, B(\mu_B|\mu),\, S(\mu_S|\mu),\, \Phi_i(\cdot|\mu_i, \mu)\Big)_{i\in\{B,S\}}$$

such that the beginning of period exit strategies $\chi_i(\cdot|\mu_i)$ are optimal given the expected bargaining payoffs, the market continuation values $W_i(\cdot|\mu_i)$ satisfy the Bellman Eqs. (1) and (2), and the buyer and seller stocks $B(\mu_B|\mu)$, $S(\mu_S|\mu)$, together with the distributions of active trader types $\Phi_i(\cdot|\mu_i, \mu)$ satisfy the steady state conditions (5)–(8). Moreover, the Fundamental Imbalance condition must hold, and the beginning of period beliefs are updated from optimistic to pessimistic if a trader is not matched in the previous period.

### 2.2 Bargaining game

In this subsection, we derive a *Perfect Bayesian Nash Equilibrium* (PBNE) of the bargaining game played in a given meeting. The buyers and sellers who are matched and play the bargaining game view the search equilibrium $\mathcal{S}$ as *exogenously given*, which affects their payoffs in the bargaining game.

The trader's proposing strategy given his belief $\mu$ is denoted as $p_i(\cdot|\mu_i)$, and the responding strategy is denoted as $a_i(\cdot|\mu_i) \in \{0, 1\}$, where 1 indicates acceptance and 0 indicates rejection. We begin by characterizing PBNE responding strategies. Since traders have point beliefs, we assume these beliefs are not updated in the bargaining game following any price offer $p$. Then it is optimal for the buyer to accept price $p$ if $p \leq v - R_\tau W_B(v|\mu_B)$ and it is optimal to reject otherwise. In parallel, a seller will accept any price $p \geq c + R_\tau W_S(c|\mu_i)$ and will reject otherwise. Formally, we have

$$a_B(v, p|\mu_B) = \begin{cases} 1, & p \le v - R_\tau W_B(v|\mu_B), \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

$$a_S(c, p|\mu_S) = \begin{cases} 1, & p \ge c + R_\tau W_S(c|\mu_S), \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

The proposing strategies are chosen optimally by the traders given the responding strategies and their beliefs. Let $\pi_i(\hat{\mu}|\mu_i)$ be the belief trader $i \in \{B, S\}$ assigns to the belief $\hat{\mu} \in \{L, H\}$ of his bargaining partner,

$$\pi_B(\mu_S|\mu_B) = \frac{S(\mu_S|\mu_B)}{S(\mu_B)}, \qquad \pi_S(\mu_B|\mu_S) = \frac{B(\mu_B|\mu_S)}{B(\mu_S)}.$$

The key here is that the traders' beliefs $\mu_B$ or $\mu_S$ are substituted for the true state $\mu$. The seller will propose a price $p_S(c|\mu_S)$ that will maximize his expected payoff, over and above the market continuation value $R_\tau W_S(c|\mu_S)$,

$$p_S(c|\mu_S) \in \arg\max_{p \ge 0}(p - c - R_\tau W_S(c|\mu_S)) \sum_{\mu_B = H,L} \pi_S(\mu_B|\mu_S) \int_{\{v: a_B(v,\mu_B)=1\}} d\Phi_B(v|\mu_B, \mu_S), \tag{11}$$

while the proposing buyer will choose $p_B(v|\mu_B)$ to maximize a parallel expression,

$$p_B(v|\mu_B) \in \arg\max_{p \ge 0}(v - R_\tau W_B(v|\mu_B) - p) \sum_{\mu_S = H,L} \pi_B(\mu_S|\mu_B) \int_{\{c: a_S(c,\mu_S)=1\}} d\Phi_S(c|\mu_S, \mu_B). \tag{12}$$

The responding and proposing strategies given by (11) and (12) constitute a PBNE of the bargaining game $\mathcal{S}$.

**Definition 2** (*Bargaining equilibrium*) A Perfect Bayesian equilibrium of the bargaining game is defined as

$$\mathcal{B} = \Big(a_i(\cdot|\mu_i),\, p_i(\cdot|\mu_i)\Big)_{i \in \{B,S\}}$$

where the responding strategies $a_i(\cdot|\mu_i)$ are given by (9) and (10), and the proposing strategies $p_i(\cdot|\mu_i)$ are given by (11) and (12).

The strategies in the bargaining game determine the subjective *expected payoffs* $U_B(v|\mu_B)$ and $U_S(c|\mu_S)$ in the bargaining game, over and above the market continuation values $W_B(v|\mu_B)$ and $W_S(c|\mu_S)$; we only provide the expression for $U_B(v|\mu_B)$:

$$
\begin{aligned}
U_B(v|\mu_B) = \frac{1}{2}&(v - R_\tau W_B(v|\mu_B) - p_B(v|\mu_B)) \\
&\sum_{\mu_S=H,L} \pi_B(\mu_S|\mu_B) \int_{\{c:a_S(c,p_B(v|\mu_B)|\mu_S))=1\}} d\Phi_S(c|\mu_S,\mu_B) \\
&+\frac{1}{2}\sum_{\mu_S=H,L} \pi_B(\mu_S|\mu_B) \\
&\times \int_{\{c:a_B(v,p_S(c|\mu_S)|\mu_B)=1\}} (v - R_\tau W_B(v|\mu_B) \\
&-p_S(c|\mu_S))d\Phi_S(c|\mu_S,\mu_B).
\end{aligned} \tag{13}
$$

They also determine the expected trading probabilities. Recall that the trading probability $q_B(v|\mu_S, \mu_B)$ was defined the probability that a type $v$ buyer who holds belief $\mu_B$ will trade with a seller holding belief $\mu_S$, conditionally on being matched, and the trading probability $q_S(x|\mu_B, \mu_S)$ is defined in parallel. These probabilities are determined using the responding strategies in the bargaining game as follows; again, we only give the expression for $q_B(v|\mu_S, \mu_B)$:

$$
\begin{aligned}
q_B(v|\mu_S,\mu_B) = \frac{1}{2} &\int_{\{c:a_S(c,p_B(v|\mu_B)|\mu_S))=1\}} d\Phi_S(c|\mu_S,\mu_B) \\
&+\frac{1}{2}\int_{\{c:a_B(v,p_S(c|\mu_S)|\mu_B)=1\}} d\Phi_S(c|\mu_S,\mu_B).
\end{aligned} \tag{14}
$$

With the search equilibrium $\mathcal{S}$ and the bargaining equilibrium $\mathcal{B}$ being defined, we now can define a *market equilibrium* $\mathcal{E}$ as follows.

**Definition 3** (*Market equilibrium*) A market equilibrium $\mathcal{E}$ is defined as a tuple $\mathcal{E} = (\mathcal{S}, \mathcal{B})$ such that $\mathcal{B}$ is a PBNE equilibrium of the bargaining game given $\mathcal{S}$, while simultaneously, $\mathcal{S}$ is a Perfect Bayesian search equilibrium given $\mathcal{B}$.

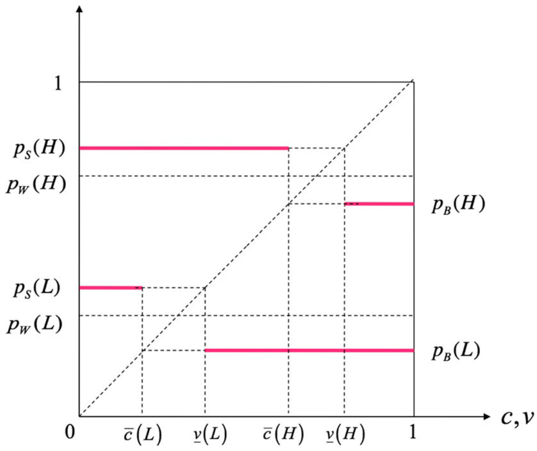
We further restrict attention to market equilibria with the following *full trade* property: the traders who share the same belief trade with probability 1 in a meeting, while the traders who have the opposite beliefs, do not trade. Formally, we have

$$
\hat{q}_B(v|\mu_S,\mu_B) = \begin{cases} 1, & \text{if } \mu_B = \mu_S \\ 0, & \text{otherwise,} \end{cases} \qquad v \in A_B,
$$

$$
\hat{q}_S(c|\mu_B,\mu_S) = \begin{cases} 1, & \text{if } \mu_S = \mu_B \\ 0, & \text{otherwise,} \end{cases} \qquad c \in A_S.
$$

We call such equilibria *optimistic search equilibria*. The structure of such equilibria is greatly simplified. which allows us to obtain our existence and convergence results in the next section. It will have the following additional properties, which will be verified in the next section. Refer to Fig. 2.

**Fig. 2** Proposing strategies in optimistic equilibrium




(1) *Participation* The sets of active buyer and seller types are intervals,

$$A_B(\mu_B) = \left[\underline{v}(\mu_B), 1\right], \qquad A_S = [0, \bar{c}(\mu_S)],$$

with

$$\underline{v}(L) > \bar{c}(L), \quad \underline{v}(H) > \bar{c}(H) \tag{15}$$

(2) *Proposing strategies* The prices are chosen equal to the marginal participating type of the partner who shares the same belief about the state:

$$p_B(\mu_B) = \bar{c}(\mu_B), \quad v \in \left[\underline{v}(\mu_B), 1\right], \tag{16}$$

$$p_S(\mu_S) = \underline{v}(\mu_S), \quad c \in [0, \bar{c}(\mu_S)]. \tag{17}$$

(3) *Responding strategies* The active traders choose to accept offers that are made to them by partners who share the same (correct) belief about the state $\mu$. The traders choose not to accept offers made by partners who have differing beliefs. This implies that the market continuation values $W_i(\cdot|\mu_i)$ are linear functions and an active type $v$ buyer with belief $\mu_B$ will accept any price less or equal than

$$v - R_\tau W_B(v|\mu_B) = \frac{R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} \underline{v}(\mu_B) + (1 - R_\tau)\, v}{1 - R_\tau + R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\}},$$

whereas an active $c$ type seller with belief $\mu_S$ will accept any price greater or equal than

$$c + R_\tau W_S(c|\mu_S) = \frac{R_\tau \min\left\{1, \frac{B(\mu_S)}{S(\mu_S)}\right\} \bar{c}(\mu_S) + (1 - R_\tau)\, c}{1 - R_\tau + R_\tau \min\left\{1, \frac{B(\mu_S)}{S(\mu_S)}\right\}}.$$

The existence of an optimistic search equilibrium will be shown in the next section, for small $\tau$. Here, we provide some intuition. Property (1) will follow from the fact that the market continuation values $W_i(\cdot|\mu_i)$ are increasing functions of trader types. Properties (2) and (3) will follow once we show that the *reservation values* of buyers and sellers, $v - R_\tau W_B(v|\mu_B)$ and $c + R_\tau W_S(c|\mu_S)$, have vanishing slopes as $\tau \to 0$. This will imply property (2), namely that the traders will choose to make offers at the marginal participating partner types $\underline{v}(\mu)$ and $\underline{c}(\mu)$. As the responding strategies are governed by the reservation values according to (9) and (10), these offers will be immediately accepted by the partners who share the same (and correct) belief, and rejected by the partners who have a differing (and wrong) belief about the state, thus verifying (3).

We now provide the intuition for the Fundamental Imbalance condition, which is crucial to our belief updating. In our equilibrium, the traders who are able to trade (i.e. share the same belief), will have the same ex-post profit equal to $(1/2)(\underline{v}(\mu) - \underline{c}(\mu))$. As their per period participation costs are also the same, $\kappa\tau$, it follows that their probabilities of meeting a suitable partner should also be the same. Therefore the optimistic and pessimistic traders should match one to one, which implies that their market stocks must be the same in any state $\mu \in \{H, L\}$. But since traders arrive with optimistic beliefs, there is also a positive stock of traders with optimistic beliefs in any state. This implies that the optimistic traders with correct beliefs are on the short side of the market.

## 3 Analysis and results

The marginal participating types of buyers and sellers must be indifferent between exiting or not. As they only make a surplus when they propose, and will only trade if their partner shares the same belief about the state. Suppose $\mu = H$. Then their indifference conditions are given by:

$$\frac{S(H)}{B(H)} \frac{1}{2} \left(\underline{v}(H) - \bar{c}(H)\right) = \tau\kappa. \tag{18}$$

for the buyers, and

$$\frac{B(H|H)}{B(H)} \frac{1}{2} \left(\underline{v}(H) - \bar{c}(H)\right) = \tau\kappa \tag{19}$$

for the sellers. Similar conditions can be written when the state is $L$.[5]

These conditions can be further explained as follows. For example, consider buyers. The marginal buyers meet sellers with probability $S(H)/B(H)$ and with probability $1/2$ offer $\bar{c}(H)$, which is accepted by any active seller they meet. Their expected profit from a meeting is just sufficient to cover their participation cost $\tau\kappa$ incurred over a period.

[5] These conditions parallel those in Shneyerov and Wong (2010b, 2011) under no aggregate uncertainty.

We now show these indifference conditions, together with the steady-state flow conditions, derived below, that determine the relevant stocks, uniquely pin down an optimistic equilibrium *candidate.*

Continuing to assume that $\mu = H$, the stock of pessimistic sellers is 0. There are three relevant stocks of buyers.

1. $B^0(L|H)$, the stock of optimistic buyers with $v \in \left[\underline{v}(L), \underline{v}(H)\right]$, who will exit voluntarily after not being matched
2. $B^1(L|H)$, the stock of optimistic buyers with $v \in \left[\underline{v}(H), 1\right]$, who will only exit through trade, once again after becoming pessimistic
3. $B(H|H)$, the stock of pessimistic buyers, who exit the market only by trading

The following mass balance conditions must hold for the stocks of buyers in a steady state optimistic equilibrium. First, the following mass balance equations must hold:

$$1 - G_B\left(\underline{v}(H)|H\right) = G_S\left(\bar{c}(H)|H\right), \tag{20}$$

This is because the traders on the shorter side of the market can only exit through trade, and only traders who share the same belief about $\mu$, trade. For example, when the state is $\mu = H$, the outgoing flow of sellers is $G_S(\bar{c}(H)|H)$ each period. Since traders leave in matched pairs, it is equal to the outgoing flow of pessimistic buyers. But the incoming flow into the stock of pessimistic buyers is equal to $1 - G_B\left(\underline{v}(H)|H\right)$. The balance of these flows in a steady state implies (20).

Next, we must have

$$\tau \cdot \left[1 - G_B\left(\underline{v}(H)|H\right)\right] = (1 - \frac{S(H)}{B(H)})B^1(L|H), \tag{21}$$

$$\tau \cdot \left[G_B\left(\underline{v}(H)|H\right) - G_B\left(\underline{v}(L)|H\right)\right] = \left(1 - \frac{S(H)}{B(H)}\right)B^0(L|H), \tag{22}$$

$$\left(1 - \frac{S(H)}{B(H)}\right)B^1(L|H) = \frac{S(H)}{B(H)}B(H|H), \tag{23}$$

*and*

$$B(H|H) + B^1(L|H) + B^0(L|H) = B(H). \tag{24}$$

Refer to Fig. 3. Equation (21) above states that the inflowing mass of buyers with $v \in \left[\underline{v}(H), 1\right]$ in a given period is equal to the outflowing mass of optimistic buyers with $v \in \left[\underline{v}(H), 1\right]$ in the market who change their beliefs to pessimistic ones upon not meeting a seller, which happens with probability $1 - \frac{S(H)}{B(H)}$. Equation (22) is a parallel statement for the inflowing mass of buyers with $v \in \left[\underline{v}(L), \underline{v}(H)\right]$, which is equal to the outflowing mass of buyers with $v \in \left[\underline{v}(L), \underline{v}(H)\right]$ who have chosen to exit the market immediately once unmatched. Equation (23) states that the inflowing mass of pessimistic buyers is equal to the mass of buyers that leaves the market through trading, which happens with probability $\frac{S(H)}{B(H)}$. Equation (24) simply re-iterates the fact that the total steady-state stock of buyers $B(H)$ is comprised of $B^0(L|H)$, $B^1(L|H)$ and $B(H|H)$.

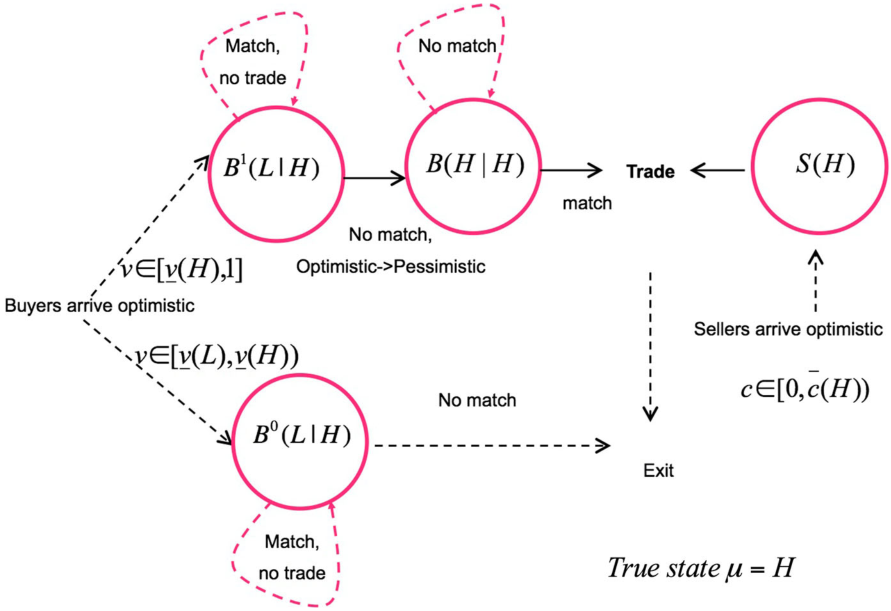



**Fig. 3** Steady-state flows

It is convenient to define a function

$$\phi(z|\mu) \equiv G_S^{-1}[1 - G_B(z|\mu)|\mu]$$

that gives the marginal seller's type given the marginal buyer's type $z$ in a steady state of the market. In particular, the equilibrium marginal types are related as

$$\bar{c}(\mu) = \phi(\underline{v}(\mu)|\mu).$$

The function $\phi\,(\cdot|\mu)$ is decreasing, and satisfies $\phi\,(0|\mu) = 1$, $\phi\,(1|\mu) = 0$.

The indifference Eqs. (18) and (19), together with four steady state conditions (21)–(24) for trader stocks and the mass balance Eq. (20) give us a system of 7 characterizing equations. Parallel equations can be written when the state of the market is $\mu = L$, where the definitions of the seller stocks $S^0\,(H|L)\,,\,S^1\,(H|L)\,,\,S\,(L|L)$ mirror those of $B^0\,(L|H)\,,\,B^1\,(L|H)\,,\,B\,(H|H)$, and the counterpart of (20) is

$$1 - G_B\left(\underline{v}\,(L)\,|L\right) = G_S\left(\bar{c}\,(L)\,|L\right). \tag{25}$$

(We omit the equations for state $\mu = L$ to save on notation.) In total, this gives us a system of $2 \cdot 7 = 14$ equations for 14 unknowns

$$B\,(H)\,,\,S\,(H)\,,\,B^0\,(L|H)\,,\,B^1\,(L|H)\,,\,B\,(H|H)\,,\,\underline{v}\,(H)\,,\,\bar{c}\,(H)\,,$$
$$B\,(L)\,,\,S\,(L)\,,\,S^0\,(H|L)\,,\,S^1\,(H|L)\,,\,S\,(L|L)\,,\,\underline{v}\,(L)\,,\,\bar{c}\,(L)\,.$$

Proposition 1 below invokes The Implicit Function theorem to show the existence of a unique solution to this system. When convenient, we index objects by $\tau$.

**Proposition 1** *The system of equations characterizing a full trade equilibrium reduces to*

$$\frac{1}{2}\left(\underline{v}_\tau(H) - \phi\left(\underline{v}_\tau(H)\,|H\right)\right) = \tau\cdot\kappa\left(1+\sqrt{\frac{1-G_B\left(\underline{v}_\tau(L)\,|H\right)}{1-G_B\left(\underline{v}_\tau(H)\,|H\right)}}\right), \tag{26}$$

$$\frac{1}{2}\left(\underline{v}_\tau(L) - \phi\left(\underline{v}_\tau(L)\,|L\right)\right) = \tau\cdot\kappa\left(1+\sqrt{\frac{G_S\left(\phi\left(\underline{v}_\tau(H)\,|H\right)|L\right)}{G_S\left(\phi\left(\underline{v}_\tau(L)\,|L\right)|L\right)}}\right). \tag{27}$$

*There exists a unique solution* $(\underline{v}_\tau(H), \underline{v}_\tau(L))$ *for all sufficiently small* $\tau \geq 0$. *Moreover, as* $\tau \to 0$, *for* $\mu \in \{H, L\}$,

$$\underline{v}_\tau(\mu) = p_W(\mu) + O(\tau), \tag{28}$$

$$\bar{c}_\tau(\mu) = p_W(\mu) - O(\tau), \tag{29}$$

*and all trader stocks are asymptotically proportionate to* $\tau$[6]*:*

$$B_\tau(H), S_\tau(H), B_\tau^0(L|H), B_\tau^1(L|H), B_\tau(H|H) \asymp \tau,$$
$$B_\tau(L), S_\tau(L), S_\tau^0(H|L), S_\tau^1(H|L), S_\tau(L|L) \asymp \tau.$$

We now show that, if $r$, $\tau$ are sufficiently small, the identified equilibrium candidate is, in fact, an optimistic equilibrium.

**Proposition 2** (Existence of Optimistic Equilibrium) *There exist* $\bar{r}, \bar{\tau} > 0$ *such that a unique optimistic equilibrium exists for* $(r, \tau) \in [0, \bar{r}] \times (0, \bar{\tau}]$.

*Proof* The proof of this proposition proceeds in several steps. The Fundamental Imbalance condition is immediate because, in state $\mu = H$, dividing the buyer's entry indifference condition (18) by the seller's (19), we obtain

$$\begin{aligned} S(H) &= B(H|H) \\ &< B(H), \end{aligned} \tag{30}$$

and in parallel, when the state is $\mu = L$, $B(L) < S(L)$.

We now need to establish the properties (1)–(3) in of the optimistic search equilibrium. A direct calculation shows that the market continuation values are equal to

$$W_B(v|\mu_B) = \frac{\min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\}}{1 - R_\tau + R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\}}\left(v - \underline{v}(\mu_B)\right), \tag{31}$$

[6] For any real-valued function $x_\tau : \mathbb{R}_+ \to \mathbb{R}_+$, we say that $x_\tau$ is *asymptotically proportionate* to $\tau$, and write $x_\tau \asymp \tau$ if, as $\tau \to 0$, the ratio $x_\tau/\tau$ is bounded away from both 0 and infinity, i.e. $lim_{\tau\to 0} inf\, x_\tau/\tau > 0$ and $lim_{\tau\to 0}\, sup\, x_\tau/\tau < \infty$.

$$W_S(c|\mu_S) = \frac{\min\left\{1, \frac{B(\mu_S)}{S(\mu_S)}\right\}}{1 - R_\tau + R_\tau \min\left\{1, \frac{B(\mu_S)}{S(\mu_S)}\right\}} (\bar{c}(\mu_S) - c). \tag{32}$$

Also notice that Eqs. (31) and (32) imply that the values of search, $W_B(\cdot|\mu_B)$ and $W_S(\cdot|\mu_S)$, are, respectively, increasing and decreasing linear functions. It follows that only the buyers with $v \in [\underline{v}(\mu_B), 1]$ and sellers with $c \in [0, \bar{c}(\mu_S)]$, will choose to participate in search.[7] This verifies property (2) of the optimistic search equilibrium.

Next, we show that profitable trade is impossible even between type 0 optimistic seller and type 1 optimistic buyer: for sufficiently small $\tau > 0$,

$$R_\tau [W_B(1|L) + W_S(0|H)] > 1.$$

Indeed, Proposition 1 implies

$$\lim_{\tau \to 0} \frac{S(H)}{B(H)}, \quad \lim_{\tau \to 0} \frac{B(L)}{S(L)} \in (0, 1). \tag{33}$$

Since the marginal types converge to the Walrasian prices according to (28) and (29), (31) and (32) imply

$$\lim_{\tau \to 0} R_\tau [W_B(1|L) + W_S(0|H)] = 1 + p_W(H) - p_W(L) > 1.$$

It also follows that the price offers $p_L(\mu_B), p_S(\mu_S)$ are acceptable, respectively, to all sellers with $c \in [0, \bar{c}(\mu_B)]$ and all buyers with $v \in [\underline{v}(\mu_S)]$. This verifies property (3), and also partially verifies (2) in that buyers and sellers will never choose to make price offers that would be acceptable to optimistic partners.

To verify the remaining part of (2), we need to show that buyers (resp., sellers) do not have an incentive to deviate by making offers below $\bar{c}(\mu_B)$ (rep., above $\underline{v}(\mu_S)$). This step requires a somewhat lengthy argument that is given in Lemma 1 in the Appendix. To gain intuition, let $\mu = H$, and consider such a deviation by the sellers. The *reservation values* of *buyers* in the bargaining game are given by

$$\begin{aligned}\tilde{v}(v|\mu_B) &= v - R_\tau W_B(v|\mu_B) \\ &= \frac{R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} \underline{v}(\mu_B) + (1 - R_\tau) v}{1 - R_\tau + R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\}}.\end{aligned} \tag{34}$$

(A parallel expression can be obtained for the sellers' reservation values $\tilde{c}(c|\mu_S) \equiv c + R_\tau W_S(c|\mu_S)$.) By Proposition 1, the perceived meeting probability

$$\min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} = \begin{cases} \frac{S_\tau(H)}{B_\tau(H)}, & \mu_B = H \\ 1, & \mu_B = L \end{cases}$$

[7] We assume that, whenever indifferent, traders choose to participate.

remains bounded away from 0 as $\tau \to 0$. Therefore, the reservation values become progressively flatter and in the limit become exactly horizontal. In addition, the same Proposition 1 implies that the gaps between the marginal participating types converges to 0. This, in turn, implies that the buyers face diminishing incentives to reduce their price offers, because the probability of these offers being rejected increases. Similarly, the sellers face diminishing incentives to increase their price offers. However, the "no deviation" result is not immediate since the profit at stake for the marginal types, equal to the difference between the seller's and buyer's marginal type, is proportionate to $\tau$ and therefore also becomes small. The proof of Lemma 1 shows that, still, the first effect dominates for sufficiently small $\tau > 0$ *and* $r \geq 0$. *Q.E.D.*

We now show that the optimistic equilibrium converges to efficiency as $\tau \to 0$. Let the *true* utilities of buyers with beliefs $\mu_B$ and sellers with beliefs $\mu_S$ in state $\mu$ be denoted as $w_B(v|\mu_B, \mu)$, $w_S(c|\mu_S, \mu)$. The targets for their convergence are the *Walrasian utilities* of the traders in state $\mu$:

$$W_B^*(v|\mu) \equiv \max\{v - p_W(\mu), 0\}, \qquad W_S^*(c|\mu) \equiv \max\{p_W(\mu) - c, 0\}.$$

In a frictional market ($\tau > 0$), there is an unavoidable utility loss due to costly search and discounting, and buyers and sellers will realize smaller utilities. Because the traders in the entering cohorts are optimistic, we only need to demonstrate convergence of the utilities of the optimistic traders. The optimistic buyers or sellers with correct beliefs about the true state will have their market utilities equal to the subjective utilities,

$$w_B(v|L, L) = \max\{W_B(v|L), 0\}, \quad w_S(c|H, H) = \max\{W_S(c|H), 0\},$$

while the ones with wrong beliefs have their true utilities determined from the recursive equations

$$\begin{aligned} w_B(v|L, H) &= R_\tau \left(1 - \frac{S(H)}{B(H)}\right) \max\{W_B(v|H), 0\} \\ &\quad + R_\tau \frac{S(H)}{B(H)} w_B(v|L, H) - \kappa\tau, \\ &= \frac{R_\tau \left(1 - \frac{S(H)}{B(H)}\right) \max\{W_B(v|H), 0\} - \kappa\tau}{1 - R_\tau \frac{S(H)}{B(H)}}, \end{aligned} \tag{35}$$

$$\begin{aligned} w_S(c|H, L) &= R_\tau \left(1 - \frac{B(L)}{S(L)}\right) \max\{W_S(c|L), 0\} \\ &\quad + R_\tau \frac{B(L)}{S(L)} w_S(c|H, L) - \kappa\tau \\ &= \frac{R_\tau \left(1 - \frac{B(L)}{S(L)}\right) \max\{W_S(c|L), 0\} - \kappa\tau}{1 - R_\tau \frac{B(L)}{S(L)}} \end{aligned} \tag{36}$$

The intuition for (35) and (36) is that, first, optimistic traders with wrong beliefs do not trade in any meeting, and second, they learn the true state $\mu$ when they do not meet a partner in the present period. The latter event occurs with with probability $1 - \frac{S(H)}{B(H)}$ for buyers when $\mu = H$ and with probability $1 - \frac{B(L)}{S(L)}$ for sellers when $\mu = L$, and then the true market utilities coincide with the believed ones. □

**Proposition 3** (Convergence to Efficiency) *The market utilities of the entering traders* $w_{B\tau}(v|L, \mu)$ *and* $w_{S\tau}(c|H, \mu)$ *converge to the utilities that traders would realize under perfect competition,*

$$\lim_{\tau \to 0} w_{B\tau}(v|L, \mu) = W_B^*(v|\mu), \tag{37}$$

$$\lim_{\tau \to 0} w_{S\tau}(c|H, \mu) = W_S^*(c|\mu). \tag{38}$$

*Proof* Without loss of generality, assume that the true state is $H$. Then the sellers' true utilities $w_{S\tau}(c|H)$ coincide with the their perceived utilities $\max\{W_{S\tau}(c|H), 0\}$, and taking the limit in (32) as $\tau \to 0$, in view of (29) and (33), implies (38). Also, the same logic applied to (31) implies that the pessimistic buyers' utilities converge:

$$\lim_{\tau \to 0} \max\{W_{B\tau}(v|H), 0\} = W_B^*(v|\mu).$$

Now, learning the true state by the optimistic buyers does not take too long. Indeed, taking the limit in as $\tau \to 0$ and using (33) again, we see that

$$\lim_{\tau \to 0} w_B(v|L, H) = \lim_{\tau \to 0} \max\{W_{B\tau}(v|H), 0\}, = W_B^*(v|\mu).$$

*Q.E.D.* □

## 4 Conclusions

We believe our work could be extended in a number of directions. Perhaps the most natural extension would involve a more general form of the prior beliefs. We have assumed extreme priors, putting all the weight on either high or low statures. This is clearly a strong assumption. A more realistic prior would put some weight on both states. Intuitively, one might expect that most of our results would go through if we assume that traders put sufficiently high probability on their most favourable state. However, developing a formal model along these lines is challenging.

The main difficulty with non-extreme priors is with the existence of a full-trade optimistic equilibrium, which is crucial for all our results. The problem is the complicated nature of beliefs off the (full trade) equilibrium path following the rejection of an offer, which affects the reservation values on the equilibrium path. This problem arises even in a simpler model with no type heterogeneity. It has so far not yielded to our efforts, but is nonetheless worth pursuing. If successful, such a model would also allow the matching probability to be less than one, which would lead to a more realistic delayed learning. We leave these extensions for future work.

## 5 Appendix

*Proof of Proposition 1* Equations (18) and (19) imply that in a full trade equilibrium, the stock of pessimistic buyers is equal to the stock of sellers,

$$B\left(H|H\right) = S\left(H\right). \tag{39}$$

Equations (21) and (22) imply

$$\begin{aligned} B^0\left(L|H\right) &= \frac{G_B\left(\underline{v}\left(H\right)|H\right) - G_B\left(\underline{v}\left(L\right)|H\right)}{1 - G_B\left(\underline{v}\left(H\right), H\right)} B^1\left(L|H\right) \\ &= \beta \cdot B^1\left(L|H\right) \end{aligned} \tag{40}$$

where

$$\beta \equiv \frac{G_B\left(\underline{v}\left(H\right)|H\right) - G_B\left(\underline{v}\left(L\right)|H\right)}{1 - G_B\left(\underline{v}\left(H\right)|H\right)} > 0.$$

Equation (23) is equivalent to

$$\left(B^1\left(L|H\right) + B^0\left(L|H\right)\right) B^1\left(L|H\right) = B\left(H|H\right)^2,$$

which upon the substitution of (40) for $B^0\left(L|H\right)$ can be solved for $B\left(H|H\right)$,

$$B\left(H|H\right) = \left(1+\beta\right)^{1/2} B^1\left(L|H\right). \tag{41}$$

Substituting (40) and (41) into (21) gives us the solution for $B^1\left(L|H\right)$ and

$$B^1\left(L|H\right) = \tau \cdot \left[1 - G_B\left(\underline{v}\left(H\right), H\right)\right] \frac{1+\beta+\left(1+\beta\right)^{1/2}}{1+\beta}, \tag{42}$$

and the other stocks $B^0\left(L|H\right)$, $B\left(H|H\right)$ are then determined from (40) and (41). The probability of meeting a pessimistic buyer is

$$\theta_B(H|H) = \frac{B(H|H)}{B(H)}$$
$$= \frac{(1+\beta)^{1/2}}{1+\beta+(1+\beta)^{1/2}}$$
$$= \left(1 + \sqrt{\frac{1-G_B\left(\underline{v}(L)|H\right)}{1-G_B\left(\underline{v}(H)|H\right)}}\right)^{-1}.$$

The entry equation, say (19) in $\mu = H$ is then equivalent to

$$\frac{1}{2}\left(\underline{v}(H) - \bar{c}(H)\right) = \tau \cdot \kappa \left(1 + \sqrt{\frac{1-G_B\left(\underline{v}(L)|H\right)}{1-G_B\left(\underline{v}(H)|H\right)}}\right). \tag{43}$$

For $\mu = L$ we obtain in parallel

$$\frac{1}{2}\left(\underline{v}(L) - \bar{c}(L)\right) = \tau \cdot \kappa \left(1 + \sqrt{\frac{G_S(\bar{c}(H)|L)}{G_S(\bar{c}(L)|L)}}\right). \tag{44}$$

The marginal types must also satisfy the mass balance conditions (20) and (25), and for $\mu \in \{H, L\}$,

$$p_W(\mu) \in \left[\underline{v}(\mu), \bar{c}(\mu)\right]. \tag{45}$$

Equations (43) and (44), together with the mass balance conditions (20) and (25), form a system of four equations for 4 unknowns, now denoted as $\left(\underline{v}_\tau(H), \bar{c}_\tau(H), \underline{v}_\tau(L), \bar{c}_\tau(L)\right)$. For $\tau = 0$, these equations imply

$$\underline{v}_0(\mu) = \bar{c}_0(\mu) = p_W(\mu).$$

The Implicit Function Theorem implies that $\bar{\tau} > 0$ exists such that a solution exists for all $\tau \in [0, \bar{\tau}]$ provided the Jacobian of this system is nonzero. Moreover, as $\tau \to 0$, the marginal types converge to the corresponding Walrasian prices, $\underline{v}(H), \bar{c}(H) \to p_W(H)$ and $\underline{v}(L), \bar{c}(L) \to p_W(L)$.

To evaluate the Jacobian, it is convenient to reduce this system by eliminating $\bar{c}(\mu)$ from Eqs. (20) and (25):

$$\bar{c}(\mu) = G_S^{-1}\left(1 - G_B\left(\underline{v}(\mu)|\mu\right)|\mu\right)$$
$$\equiv \phi\left(\underline{v}(\mu)|\mu\right),$$

where the mapping $\phi(\cdot|\mu) : [p_W(\mu), 1] \to [0, p_W(\mu)]$ (smoothly extended to an open $\varepsilon$ neighborhood of $p_W(\mu)$) has the derivative at $p_W(\mu)$ equal to

$$\phi'(p_W(\mu)|\mu) = -\frac{g_B(p_W(\mu)|\mu)}{g_S(p_W(\mu)|\mu)} < 0. \tag{46}$$

Now the system of Equations for $(\underline{v}(H), \underline{v}(L))$ becomes

$$\frac{1}{2}\left(\underline{v}(H) - \phi\left(\underline{v}(H)|H\right)\right) - \tau \cdot \kappa \left(1 + \sqrt{\frac{1 - G_B\left(\underline{v}(L)|H\right)}{1 - G_B\left(\underline{v}(H)|H\right)}}\right) = 0, \tag{47}$$

$$\frac{1}{2}\left(\underline{v}(L) - \phi\left(\underline{v}(L)|L\right)\right) - \tau \cdot \kappa \left(1 + \sqrt{\frac{G_S\left(\phi\left(\underline{v}(H)|H\right)|L\right)}{G_S\left(\phi\left(\underline{v}(L)|L\right)|L\right)}}\right) = 0. \tag{48}$$

The Jacobian of this system at $\tau = 0$ is

$$\begin{aligned}
&\begin{vmatrix} \frac{1}{2}\left(1 - \phi'(p_W(H)|H)\right) & 0 \\ 0 & \frac{1}{2}\left(1 - \phi'(p_W(L)|L)\right) \end{vmatrix} \\
&= \frac{1}{4}\left(1 - \phi'(p_W(H)|H)\right)\left(1 - \phi'(p_W(L)|L)\right) \\
&> 0,
\end{aligned}$$

where the inequality in the last line follows from (46). *Q. E. D.* □

**Lemma 1** *A threshold $\bar{r} > 0$ exists such that, for all $r \in [0, \bar{r}]$ and $\tau < 1$, buyers do not have an incentive to deviate by offering prices p that are* not *acceptable to some active sellers who share the same belief about $\mu$:*

$$p \in \left[\tilde{c}(0|\mu_B), \bar{c}(\mu_B)\right).$$

*Likewise, sellers do not have an incentive to deviate by offering prices p that are* not *acceptable to some of the buyers who share the same belief about $\mu$:*

$$p \in \left(\underline{v}(\mu_S), \tilde{v}(1|\mu_S)\right].$$

*Proof* Without loss of generality, let's assume that the state is $\mu = H$ and focus on the seller. Whenever the corresponding stocks are positive we denote the distribution of active buyer types with belief $\mu_B$ when the true state is $\mu$ as $\Phi(\cdot|\mu_B, \mu)$, while the distributions of their reservation values $\tilde{v}(v|\mu_B)$ are denoted as $\tilde{\Phi}(\cdot|\mu_B, \mu)$.

For notational expedience, from now on we denote the probability that a buyer with belief $\mu_B = H$ will meet a seller as

$$\ell_B^* \equiv \frac{S(H)}{B(H)} = \frac{B(H|H)}{B(H)},$$

where the second equality follows from (30).

The proposing strategies must be non-decreasing by standard single-crossing arguments; the proof parallels Lemma 2 in Shneyerov and Wong (2010b) and is omitted. It is therefore sufficient to show that the *marginal* types will not deviate. First, we focus on the incentives of the sellers (a symmetric argument will apply for the buyers, with obvious changes). The expected profit contingent on proposing $\lambda \geq \underline{v}(\mu)$ is

$$\pi_S\left(\bar{c}\left(\mu\right),\lambda|\mu\right)=\left(\lambda-\bar{c}\left(\mu\right)\right)\left(1-\tilde{\Phi}\left(\lambda|\mu,\mu\right)\right),$$

and its slope is

$$\begin{aligned}\frac{\partial\pi_S\left(\bar{c}\left(\mu\right),\lambda|\mu\right)}{\partial\lambda}&=\left(1-\tilde{\Phi}\left(\lambda|\mu,\mu\right)\right)-\left(\lambda-\bar{c}\left(\mu\right)\right)\tilde{\Phi}'\left(\lambda|\mu,\mu\right)\\&=-\tilde{\Phi}'\left(\lambda|\mu,\mu\right)\left[\tilde{J}_B\left(\lambda|\mu\right)-\bar{c}\left(\mu\right)\right]\end{aligned}\tag{49}$$

where $\tilde{J}_B\left(\lambda|\mu\right)$ is the "virtual type" that corresponds to the distribution of reservation values $\tilde{\Phi}\left(\cdot|\mu,\mu\right)$,

$$\tilde{J}_B\left(\lambda|\mu\right)\equiv\lambda-\frac{1-\tilde{\Phi}\left(\lambda|\mu,\mu\right)}{\tilde{\Phi}'\left(\lambda|\mu,\mu\right)}.$$

Notice that $\tilde{\Phi}\left(\lambda|\mu,\mu\right)=\Phi\left(\tilde{v}^{-1}(\lambda|\mu)|\mu,\mu\right)$. Contingent on meeting a seller, pessimistic buyers trade with probability 1 regardless of their type. Therefore, their distribution of types in the market is a truncation of the inflow distribution,

$$1-\Phi(v|\mu,\mu)=\frac{1-G_B(v|\mu)}{1-G_B(\underline{v}\left(\mu\right)|\mu))}\quad\left(v\geq\underline{v}\left(\mu\right)\right).$$

From (34), $\tilde{v}\left(v|\mu\right)$ is a linear function with the slope

$$\tilde{v}'\left(v|\mu\right)=\frac{1-R_\tau}{1-R_\tau+R_\tau\ell_B^*\left(\mu\right)}$$

Since $\tilde{v}\left(\underline{v}\left(\mu\right)|\mu\right)=\underline{v}\left(\mu\right)$, we can explicitly solve for the responding strategy,

$$\tilde{v}\left(v|\mu\right)=\frac{\left(1-R_\tau\right)v+R_\tau\ell_B^*\left(\mu\right)\underline{v}\left(\mu\right)}{\left(1-R_\tau\right)+R_\tau\ell_B^*\left(\mu\right)}.\tag{50}$$

From (34), the inverse responding strategy is

$$\tilde{v}^{-1}\left(\lambda\right)=\frac{\left(1-R_\tau\right)+R_\tau\ell_B^*}{1-R_\tau}\lambda-\frac{R_\tau\ell_B^*\underline{v}\left(\mu\right)}{1-R_\tau}.$$

Then

$$1-\tilde{\Phi}\left(\lambda|\mu,\mu\right)=\frac{1-G_B\left(\tilde{v}^{-1}\left(\lambda|\mu\right)|\mu\right)}{1-G_B\left(\underline{v}\left(\mu\right)|\mu\right)},$$

$$\begin{aligned}\tilde{\phi}\left(\lambda|\mu,\mu\right)&=\frac{d\tilde{v}^{-1}\left(\lambda|\mu\right)}{d\lambda}\frac{g_B\left(\tilde{v}^{-1}\left(\lambda|\mu\right)|\mu\right)}{1-G_B\left(\underline{v}\left(\mu\right)|\mu\right)}\\&=\frac{\left(1-R_\tau\right)+R_\tau\ell_B^*}{1-R_\tau}\frac{g_B\left(\tilde{v}^{-1}\left(\lambda|\mu\right)|\mu\right)}{1-G_B\left(\underline{v}\left(\mu\right)|\mu\right)},\end{aligned}$$

and

$$\begin{aligned}\tilde{J}_B(\lambda|\mu) &\equiv \lambda - \frac{1-\tilde{\Phi}(\lambda|\mu,\mu)}{\tilde{\phi}(\lambda|\mu,\mu)}\\ &= \lambda - \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\frac{1-G_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}\\ &= \lambda - \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\tilde{v}^{-1}(\lambda|\mu)\\ &\quad + \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\left(\tilde{v}^{-1}(\lambda|\mu) - \frac{1-G_B\left(\tilde{v}^{-1}(v|\mu)|\mu\right)}{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}\right)\\ &= \frac{R_\tau\ell_B^*\underline{v}(\mu)}{(1-R_\tau)+R_\tau\ell_B^*} + \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}\tilde{J}_B(\lambda|\mu) &= \frac{1}{(1-R_\tau)+R_\tau\ell_B^*}\\ &\quad\times\left((1-R_\tau)J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right) + R_\tau\ell_B^*\underline{v}(\mu)\right).\end{aligned}\tag{51}$$

Substituting (51) in the slope formula (49), we obtain

$$\begin{aligned}\frac{\partial\pi_S(\bar{c}(\mu),\lambda|\mu)}{\partial\lambda} &= -\tilde{\Phi}'(\lambda|\mu,\mu)\{\frac{1}{(1-R_\tau)+R_\tau\ell_B^*}\\ &\quad\times((1-R_\tau)J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right) + R_\tau\ell_B^*\underline{v}(\mu)) - \bar{c}(\mu)\}.\end{aligned}\tag{52}$$

Clearly, a deviation to $\lambda < \underline{v}(\mu)$ is not profitable, so we only need to consider $\lambda > \underline{v}(\mu)$. A necessary condition for such a deviation to be not profitable is that $\partial\pi_S(\bar{c}(\mu),\lambda|\mu)/\partial\lambda \leq 0$ at $\lambda = \underline{v}(\mu)$, i.e. the expression in the brackets on the right-hand side of Eq. (52) is non-negative when $\lambda = \underline{v}(\mu)$. This is also sufficient because of the assumed monotonicity of $J_B(\cdot|\mu)$ (Assumption 1). This gives us the inequality

$$\frac{(1-R_\tau)J_B\left(\underline{v}(\mu)|\mu\right) + R_\tau\ell_B^*\underline{v}(\mu)}{(1-R_\tau)+R_\tau\ell_B^*} - \bar{c}(\mu) \geq 0.$$

We now show that this inequality is satisfied for small $r$. We can rewrite it as

$$\underline{v}(\mu) - \bar{c}(\mu) - \frac{(1-R_\tau)}{(1-R_\tau)+R_\tau\ell_B^*}\frac{1-G_B\left(\underline{v}(\mu)|H\right)}{g_B\left(\underline{v}(\mu)|H\right)} \geq 0.\tag{53}$$

From either (18) or (19) we have $\underline{v}(\mu) - \bar{c}(\mu) = 2\tau\kappa/\ell_B^*$. Substituting this into (53) and replacing $\frac{1-G_B(\underline{v}(\mu)|H)}{g_B(\underline{v}(\mu)|H)}$ with an upper bound $1/\underline{g}_B$, and $(1-R_\tau)+R_\tau\ell_B^*$ with

$R_\tau \ell_B^*$, we have a stronger inequality that is sufficient for no deviation:

$$\frac{2\tau\kappa}{\ell_B^*} - \frac{(1-R_\tau)}{R_\tau \ell_B^*}\frac{1}{\underline{g}_B} \geq 0.$$

Alternatively,

$$\frac{1-e^{-r\tau}}{\tau e^{-r\tau}} \leq 2\kappa \underline{g}_B. \tag{54}$$

The l.h.s. of the above equation, $(e^{r\tau}-1)/\tau$, is an increasing function of $\tau$ because $e^{r\tau}-1$ is a convex, increasing function of $\tau$ taking value 0 at $\tau = 0$. Therefore, if $\tau \leq 1$, it is sufficient to require

$$e^r - 1 \leq 2\kappa \underline{g}_B,$$

or equivalently

$$r \leq \log\left(1 + 2\kappa \underline{g}_B\right).$$

A parallel argument applied to marginal sellers. *Q. E. D.* □

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